A Constructive Theory of Representation

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Abstract
A new constructive theory of representation is presented, which is based on the intuitive experiences of force fields. These force fields generate images that become elements in a higher level force field. A structure is obtained by embedding force fields in a projective geometry. Stable bipolar force-fields emerge from specific force-fields. Finally, a stable generic pattern with an orientation is derived. This generic pattern acts as infrastructure for all specific patterns, constituting a modular approach. Representation is identified as an interaction between two polar forces: the driving target force and the embodying representational force. Applications and examples abound.
Key words
Role patterns, change, force-fields, representation, modular approach, philosophy of science.

1 Introduction

The problem I intend to address is well formulized by Roman Frigg (2008): "When presenting a model scientists perform two different acts: they present a hypothetical system as the object of study, and they claim that this system represents the particular part or aspect of the world that we are interested in, the so-called the target system. In working scientists' presentations of models these two acts are rarely kept neatly separate, and most of the work goes into the first while the second, the specification of the representational relation between the hypothetical system and the target, is often left implicit." That specification is made in this paper (Definition 8), among other results of the new approach.

Frigg adds that every tenable account of model systems, which are representations of a selected part or aspect of the world, has to address several issues that have not been solved yet. I will show in the Conclusion section that my approach indeed gives satisfactory answers to these questions. This approach is based on the experience and identification of force fields in phenomena in the world. A force field describes the interaction between several forces that act upon each other. The actors may be participants in a discussion, players and spectators in a stadium, or an artist handling his material. Force fields are evidently also experienced and identified in nature. There is an abundance of force fields, some of which are directly related, others are on a far distant of a phenomenon. Since mathematics is the best available tool to create a structure in this abundance, I invoke two fundamental mathematical concepts, which are used to structure the reasoning. The first one is projective geometry, which theory de-

[1] I owe important insights from numerous discussions with my friends Rob Gilles, Frans van Doorne and Jack Vromen; I am grateful for stimulating remarks by Stephan Hartmann and Margaret Morrison on an earlier version. Willem Haemers, Ad Pikkemaat and René van den Brink made very helpful suggestions.

scribes invariant transformations. I interpret this as looking to an object from different angles and with various levels of specification. The second is matroid theory; a matroid is essentially a set with an 'independence structure' defined on it. I interpret the force-fields as being minimally dependent. The two are in the end combined on a very natural way into a tripolar role pattern. That pattern describes the interdependency between generic force fields. When endowed with an orientation – breaking the symmetry of the mathematical object – the tripolar role pattern becomes a generic roles pattern that serves as an infrastructural reference for all specific force fields. In this structure the concept of representation is identified as an interaction between two polar forces: the driving target force and the embodying representational force [2].

After describing some current theories of scientific representation in Section 2, I introduce the generic roles pattern in Section 3; the constructive approach leading to the fundamental equation of representation and the modular theory is presented in Section 4. In Section 5 I present case studies to clarify the construction of representations. The last section concludes.

2 Current theories of scientific representation; desiderata

Frigg and Hartmann (2006) present the current state of the art in scientific modeling. They put two problems at the center stage of the discussion. The first problem is to explain in virtue of what a model is a representation of something else. If we understand models as n on-linguistic entities, they write, we are faced with the new question of what it is for an object (that is not a word or a sentence) to scientifically represent a phenomenon. Until recently this question has not attracted much attention in twentieth century philosophy of science. They refer to Frigg (2006), Giere (2004), Suárez (2004), van Fraassen (2004) as recent publications having addressed this and other closely related problems. Others dismiss it as a non-issue (Callender and Cohen 2006, Teller 2001).

This controversy is today reflected in the discussions among philosophers of science about scientific representation. Is there a fit between a model and the world? What kind of fit? It cannot be perfect; since not all details can be motivated, it can't be a Truth or Falsity. What makes a model representationable?[3] Ronald Giere (2006) argues that there exists a hierarchy of representationable models, culminating in an overlapping theory. He observes that the standard Model – World conception has to be extended with intentions how to use that scientific representation. So it becomes an intentional concept: Model – Object – Purpose. The agent-based conception presents a useful framework for understanding the role of models. But these agents are embodied and embedded in multiple overlapping frames and cultures, so I think that this extension requires a more holistic approach. Chakravartty also observes a shift in emphasis to scientific representation: from the informational view (similarity; interpretation; isomorphism; entities) to the functional view (demonstration; inference; processes; capacities). Morrison, in line with Suppes (1960), aims at differentiating a theory from its models by highlighting the different notions of representation and explanation in either concept, based on the level of generality. She refers to Nancy Cartwright (1999) who states that only models represent, not the theory. Representation is just one of the criteria. She accepts the semantic

[2] The term 'representational force' has been introduced by Suárez (2004).
[3] These and other research questions were addressed in the conference *Models and Simulations 2*, organized in 2007 by the Tilburg Center for Logic and Philosophy of Science. The views of the persons mentioned without reference in this section are my interpretations of their talks at the conference.

view in that a representation is at least a structured mapping, an isomorphy, but agrees with Suarez (2003) that it is something more. A theory is more than a family of models: it adds generality. The problem is how to identify this. You need a theory to distinguish! Models are mediators in practice, and explanatory tools. Uskali Mäki, finally, sees a model as an imagined, isolated system, and refers to *der isolierter Staat* by J.H. von Thünen. The causal force in the model is true to some degree in the similarity relation.

3 Worlds of oriented patterns

3.1 Identifying Image generating Force Fields

The methodological foundation of a theory of representation may learn from mainstream theories in mathematics. The school of Logicism – promoted by Frege, Couturat and Russell – aimed at founding mathematics from logical principles only. The school of Formalism – headed by Hilbert – focused on the formal consistency of rules applies on otherwise meaningless signs and symbols. The mathematician Brouwer (1907) introduced Intuitionism as foundation for mathematics. Instead of using abstract definitions and axioms as building stones, he aimed at constructing mathematics from elementary experiences. He proposed primeval intuition about the inseparable two-unity of continuum and discrete as building stone. Consecutive discrete mental events – connected by a continuum of time – create identical sequences, which can be qualified and called by a number. So numbers are sequences of abstracted experiences. The connecting continuum between two numbers, say 0 and 1, is the experience of a 'between', which experience is identified with an inserted point, called ½ or 0.1 (mod 2). Repeating this procedure with the numbers 0 and 0.1 results in 0.01 (mod 2); in this way, the set of rationals is constructed. The set of irrationals is represented by a 'choice sequence', which is an ever growing mathematical object, a 'becoming' by construction and not a 'being'.

The crux of Brouwer's method as I see it, is to induce a separation within a continuum – or within the 'between' sets of the integers – by constructing an operation that generates a 'qualitative jump' in the discriminating criterion of the original object set. When aggregating 'consecutive discrete mental events' and qualifying the sequence, these events were elevated from their homogeneous, indiscriminate background and grouped into an event with a new property, which is: being numbered. This property – the numbers – forms a new reality abstracted from the original events. So the aggregation operation on a set of entities embodies these entities in entities of a higher level, which entities are abstracted from the lower level entities and separated according to the properties of the aggregation operation. Let me call this operation the *Brouwer embodying force field*. This operation is an element of the structure that I claim is necessary for adequately analyzing the concept of representation. In Definition 3 that structure is called an 'Oriented Interaction Pattern', containing the operation of embodying (aggregation) that is described above. The embodying operation will be qualified as a force field that generates a qualitative jump in the indexicality of elements, called an *image*.

In physics one may observe such a jump from quarks to atoms, from atoms to molecules, and from molecules to chemical elements, as examples of qualitative changes of objects resulting from the working of a (varying) *physical embodying force field*. In economics one may observe a group of individual suppliers of labor who unite to a labor union, which is a higher level object with a larger negotiating force than the forces existing at the individual level.

Again, I identify the working of an embodying field force, viz., aggregation and organization of individuals, as the generator of this qualitative jump from the original set of individuals.

In analogy, Giere (1988) has observed that representation is also produced by a representational force, which I call the *Giere embodying force field*. But to see that, we need a structure that relates this force field to possibly other force fields in the same context. That framework is constructed in the next section by applying the embodying operation on consecutive levels of abstraction. I will show in Section 5 that this method is self-reflexive in the sense that has been applied on the problem of constructing the unknown structure.

3.2 Experiencing positional patterns and positional force fields (PFF)

Patterns and positions, as well as forces and motion are closely related. If we want to describe patterns, we have to start with identifying positions from which the pattern is viewed. If we want to describe interaction or motion between these patterns, we have to identify the forces on these patterns. And if we want to describe pattern generating forces, such as representations, we have to identify the structure of these forces.

So I introduce the concept of a '***positional pattern***' as a first subjective elementary experience of some phenomenon or event in the world, indicated by a description *P*(*World*), where *World* indicates the phenomenon investigated or experienced, combined with the coordinates of the experiencing individual in terms of time, place and so forth. A deer that is thirsty experiences the pattern from its position of standing somewhere and seeing a bank of a river where it can drink. A baby experiences in her position the pattern with a desired object to which she can crawl. A firm experiences its position in the pattern of a market and moves to a more profitable position in that pattern. Positions are inseparably connected by the experience of '*motion*'. Abstracting from the contents, living organisms have the capacity of experiencing, reactivating and moving[4] a pattern.

The mathematical concept of a ***projective geometry***[5] is an appropriate tool for modeling motion in such a positional pattern. A geometry specifies the concept of a figure and calls two figures congruent if there exists a transformation that makes that figures equal. Consider, for example, the *real projective plane* and imagine a single eye located at the origin overlooking a three dimensional scene. The vision of this eye is limited in two ways, according to Henle (1997:135). First, it sees only objects within its *field of vision*, a circular cone with vertex at the eye (or an artificial eye as a telescope). Second, the eye sees only one point in each direction; that is, cannot see behind things. Therefore, for the purpose of modeling vision, all points along each ray emanating from the eye are identical because the eye sees only one of these points at a time. The two-dimensional images seen by the eye are almost congruent when the eye moves its position gradually to some other position in the three-dimensional space. Walking around a (three-dimensional) house, the individual generates a sequence of (two-dimensional, projective) positional patterns that the individual aggregates to her 'representation' of the house.

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[4] The brain of man and animals with eyes is accustomed to rotating (= moving) patterns, because the lenses in the eye inverse the real image.

[5] A projective geometry is – according to Henle (1997: 186) – a universal geometric tool in several ways. First, because the parameter spaces of important types of geometric objects tend to be projective spaces; second, because projective geometry contains many other geometries, including all metric plane geometries; third, because the coordinates of a projective geometry can be chosen from a variety of fields, so that projective geometry extends into areas of complex and discrete mathematics. See also the Appendix.

Such a 'walk' may be viewed as a sequence of positions, x^s, $s = 0,1,2,\ldots,t$, which is a dependent set of vectors if it is closed under consecutive operations on these vectors, so $x^0 \oplus x^1 \oplus \ldots \oplus x^t = 0$. This interaction between the vectors in a minimally dependent set is a specific manifestation of the *embodying force field* with the capacity of creating an *image* that is produced by the embodying force field. This image is a new quality that does not belong to the original pattern, viz., the full representation of the house from the sequence of positional patterns of the house.

A formal description of a pattern is given by the positions in an n-dimensional projective geometry, which are represented by vectors with n+1 coordinates, not all zero, in the space P_n. On these vectors is an operation $\oplus_F$ defined that describes motion in a mathematical field F. The pair $G_{n,F} = (P_n, \oplus_F)$ is an n-dimensional projective geometry over the field F. The dimension gives an indication of the number of relevant patterns relevant for generating a force field that creates an image; the operator gives an indication of the kind of interaction between these patterns; and the mathematical concept of a field gives an indication of the complexity required for an adequate description of the phenomenon. A consumer pattern, for example, parametrizes a commodity space that is ordered by a utility function or preference relation, a specification of resources, and a choice rule; this may possibly result in an excess-demand function. Let an economy be given where each consumer is characterized by a consumer pattern. The force field spanned by an embodying force – such as a market – and the individual demand patterns creates an interaction that generates a new concept on a higher level, viz., the consumer demand. The standard theoretical aggregation operation in this force field is linear addition of individual demand (see Theil, 1954), which is, however, a too simple aggregator to describe aggregation as it appears in the real world with all its external effects[6]. The purpose here is, however, not to specify some concrete force field, but to identify the relation between the various force fields.

Definition 1: Specific Positional Patterns and Force Fields; an Image

A primary Experience of a phenomenon in the World is a pattern of a series of positions, *P*(*World*), originating from a given experiencing position, which positions account for the least number that is required to generate a force field capable of creating an image of that force field. □

A formal description of a force field in a vector space with patterns as elements is a minimally dependent set of patterns (a hypergraph) in that space. This hypergraph acts as an element in the dual vector space. Similarly, the set of images created by force fields in *P*(*World*) is called the *image set of positional patterns*, *P**(*World*).

That image or embodied position may be a specific house, when the force field is walking around the house, or a typical hunter, when the force field results in an aggregate class of hunters. It is formed in time by the process of consecutive mental events of sequences experienced in similar sequences of positions that spans a force field, but it receives relative inde-

[6] It may be even worse: some macroeconomic models claim to give a picture of reality by introducing a "representative agent" instead of a mechanism that coordinates individual activities. Kirman (1992) explains that the reduction of the behavior of a group of heterogeneous agents – even if they are all themselves utility maximizers – to the behavior of one utility maximizing "representative" agent whose choices coincide with the aggregate choices of the heterogeneous individuals, is both unjustified and leads usually to misleading or wrong conclusions.

pendence from its original roots by the qualitative jump through its incorporation in a second level pattern. An experience on the second level is based on the first level experience of a position in a pattern. An image may be part of another phenomenon in the world and assume a position in that pattern, which is called the *image transformation of a force field*. The force field may be artisan or artistic and creating works of art, scientific and creating models, industrial and creating tools, social and creating clans, biological and creating bodies, or physical and creating molecules[7].

Definition 2: The Image transformation of Force Fields

An image transformation assigns to a force field in some positional pattern, *P*(*World*), an image in the positional pattern of images, *P**(*World*), generating an image pattern. The primary positional pattern in *P*(*World*) may be embedded in an n-dimensional projective geometry over the field F_1; the secondary positional pattern may be embedded in an m-dimensional projective geometry over the field F_2, where the choice of dimension and field is determined by the specifications required to identify the patterns. □

Formally, given a positional pattern described by $G_{n,F} = (P_n, \oplus_F)$ in $\Pi(n,F_1)$, then the *image transformation,* T : $\Pi(n,F_1) \to \Pi^*(m,F_1)$, assigns to a minimally dependent set (a force field) a position in its dual positional pattern (an image) in the image space $G^*_{n,F} = (P_n^*, \oplus^*_F)$ in $\Pi^*(m,F_1)$,. If the fields are identical and the image space is the dual space of hypergraphs, then both spaces are isomorphic and. The image transformation is then the standard mathematical duality operation.

3.3 Experiencing stable role patterns and stable role force-fields

A secondary experience is based on the stability of the image transformation of a force field pattern; it is a primeval intuition of a pattern of roles in a stable generation of an image.

Gilles e.a. (2008) have shown that in a relational economy, in which agents are searching for single matching patterns (the positional force fields). A generically stable matching pattern (the image transformation) in which the agents can trust the outcome, occurs if and only if the set of agents is partitioned into two subsets, e.g., demand and supply, with each subset acting in a role[8]. For such an economy, a *bipolar role force field* is a sufficient and necessary condition for generic stability, that is, stable for all profiles of individual hedonic preferences. Although this property may not be translated to other specific positional patterns, it shows that stability and bipolarity of role patterns are somewhere closely related. Other role patterns exist, of course, and may be stable[9].

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[7] A comparable procedure has been proposed by Shackle (1961). Evidence theory assumes that, for a certain individual and at a certain point in time, a provisional frame of discernment is given, and it investigates which beliefs this individual forms out of the empirical evidence he receives.

[8] A structurally similar problem of interaction between two parties has been analyzed by Durkheim (1893) and Parsons (1937). In their vision, internalized norms and values produce the desired social order and solidarity. However, no explanation of the emergence of these internalized norms and values has been provided. Why there is not a war of all against all, as Hobbes (1651) poses the question? What is the driving force in human behavior?

[9] Gilles e.a. (2008) have shown that star-shaped patterns are generically stable if cycles are excluded. These patterns provide trust patterns required for cooperation and cooperative institutions (Coleman, 1990).

A role is based on properties and characteristics that determine its position relative to other roles in the phenomenon. The interaction between roles is determined by an operator that defines the ***rules of interaction*** between roles. A set of interacting roles is again called a *force field* if these roles are minimally dependent. A role may be involved in various interactions.

A formal, but again not an exhaustive description of a role pattern is given by the positions in an *n*-dimensional projective geometry, which are represented by vectors with $m+1$ coordinates, not all zero, in the space P_m. On these vectors is an operation $\oplus$ defined that describes motion in a mathematical field F_2, which is now much simpler because only relative positions have to be indicated, which can be described qualitatively. The pair $G_{m,2} = (P_m, \oplus)$ is an *n*-dimensional projective geometry over the field F_2. The dimension gives an indication of the number of relevant role patterns for generating a force field that creates an image; the operator gives an indication of the kind of interaction between these patterns; and the mathematical concept of a field gives an indication of the complexity required for an adequate description of the phenomenon.

The parameters characterizing a coordinate in a role are given a comprehensive, qualitative meaning, as determined by the underlying pattern of specific positions. The parameter only indicates whether a specific property or capability of a role is present in a coordinate or not. That restricts the field describing the parameters to the smallest field consisting of only two elements: 0 and 1. Every role has a unique position in the space; the number of positions is equal to ($2m$+1-1). The operation $\oplus$ on these vectors is determined by coordinate-wise addition modulo 2, that is, $0 \oplus 0 = 1 \oplus 1 = 0$ and $0 \oplus 1 = 1 \oplus 0 = 1$. For example, the vectors (0, 1, 0) $\oplus$ (1, 1, 0) = (1, 0, 0). Since the zero-position is excluded, this operation assigns to each pair of different roles (positions) a unique third role (position) in the space with some characteristic property. Summarizing:

Definition 3: Specific Role Patterns and Rules
Given a pattern *P*(*World*) that is partitioned in several force fields generating specific images in *P**(*World*). The role-image of such a force field, or the ***role of an image***, is identified by sufficient qualitative characteristics (capabilities) of the images that may be present or absent in some role. These characteristics determine the relative position of a role relative to the other roles in the phenomenon of the World. The roles form a pattern, *RP*(*World*), that is experienced. They interact according to the ***rules of interaction***, which is defined by coordinate wise addition modulo 2. So roles never generate roles with identical characteristics. □

Formally, a *role pattern*, *RP*(*World*) is described in an *m*-dimensional projective geometry over the field with only two elements, {0,1}, indicated by $G_{m,2} = (P_m, \oplus)$. A role is described by a vector in P_m that has $m+1$ coordinates that are either 1 or 0, but not all 0, indicating whether certain qualitative properties are present in some role (vector) or absent. Interaction between roles is described by the operation $\oplus$ on the set of roles, defined by coordinate wise addition modulo 2; it defines the ***rule of motion*** for the roles involved. .

An example of a role pattern is an organization diagram of a firm in which the arrows connecting the roles describe the tasks and means for each pair of roles. Another example is an econometric model in which equations define dependent sets of variables. Tinbergen (1956) distinguishes four roles for the variables in a model: target variables, instrumental variables, exogenous and endogenous variables. This role pattern depends on the underlying specification of the model.

Definition 4: The Role transformation of Force Fields

A role transformation assigns to a force field in some positional pattern, *P*(*World*), a role in the positional pattern of images, *RP* (*World*), generating a role pattern. The primary positional pattern may be embedded in an n-dimensional projective geometry over the field F_1; the secondary positional pattern may be embedded in an m-dimensional projective geometry over the field F_0, where the choice of dimension and field is determined by the specifications required to identify the patterns. □

Given a first level experienced positional pattern, *P*(*World*) described by $G_{n,F} = (P_n, \oplus_F)$, the *image set of positional patterns*, *P**(*World*) is described by $G^*_{n,F} = (P_n^*, \oplus^*_F)$. The *role transformation* T_2 : $\Pi(n,F_1) \rightarrow \Pi(m,F_2)$, assigns to a subset of images a position in the role pattern, *RP*(*World*), described by $G_{m,2} = (P_m, \oplus)$. Similarly the *role-image transformation,* T_2^* : $\Pi(n,F_1) \rightarrow \Pi^*(m,F_2)$. The inverses of T_2 give a specification of the empowerment pattern required for generating an image or a force-field[10].

The embodying force may also create more force fields simultaneously. The market generates a demand force field together with a supply force field. On a more fundamental level, Gilles e.a. (2008) have shown that in a relational economy, in which agents are looking for a match with another agent, a generically stable matching pattern occurs if and only if the set of agents is partitioned into two roles. The relevant embodying force here is stabilizing matches, which outcome is basic for institutional behavior. These complementary role force fields I call the *Gilles embodying force fields*.

When applying the Gilles' embodying force on a set of people with both hunting and gathering capabilities, a stable matching pattern requires a bipartition of this set, generating roles. When these capabilities differ among people in an economy, a bipartition in hunters and gatherers appears. This generates specialization of labor: one of the basic 'laws' in classical economics. When shifting the agents' capabilities in the set of people from gathering to tool making and applying the same embodying force generates a class of hunters and a class of toolmakers. So the embodying force plays a role in the behavior of people – or any other set of entities – that surpasses the specification of these people. Similarly, the image of a class of hunters surpasses the set of individuals with – among others – hunting capabilities.

When hunters and toolmakers exchange tools and game, they communicate with each other in order to value their products and to exchange them. This is another type of force field, spanned by hunters and toolmakers who communicate, which interaction generates the exchange value or other valuable information required for their acting. Since Habermas (1981) introduced the concept of 'communicative action' that binds the norms of communication, I propose to call this the *Habermas force field*.

So a first description of intuitive experiences produces already three kinds of force fields: the embodying Giere force field, the Gilles force field and the Habermas force field. The question arises whether we can discover more such force fields and how these force fields relate. For

[10] This property of a force-field resembles the concept *shi* in the Chinese tradition; it consists in organizing circumstances in such a way as to derive profit from them (Sunzi, cited by Jullien, *The propensity of things*, 1995). The neoclassical production function is a specification in terms of commodity bundles relating inputs to outputs.

that purpose we need a structure, which is obtained by submitting the second level experiences – such as being member of a class – to the rules applied on the first level experience of a positional pattern. For example, the positional pattern of a hunter's role is a world in which other roles occur that influence his behavior. In that world, the hunter experience is embodied in a strategic knowledge common to all hunters surpassing the individual hunter. Similarly, the experience of being hunted is also embodied in a strategic knowledge common to all animals of the group being hunted. The embodying force defines the interaction between the available capabilities and the common conditions in which the deadly play of hunt – or the art of war – occurs.

Extension of a role pattern in a model is possible by (i) adding an extra dimension to the role vector, or (ii) applying an extra level transformation resulting in a higher form of incorporation. The first operation extends the model horizontally by introducing a larger number of characteristics; the second operation extends the model vertically by repeated level transformation. Although both operations are not bounded – which is a desirable property in model construction – these types of extensions are not satisfactory. The construction of ever more complex experiences leads to an infinite sequence of growing objects from plays to meta-patterns, with a patterns' complexity beyond our imagination and our intuitive experience. However, people may get a grasp on this growing complexity if they see a pattern in the roles and the rules of motion in a play that is invariant for increasing complexity. Such a unique and stable interaction pattern exists for bipolar force fields and is constructed in the next subsection.

What is required is a pattern that is: (i) invariant under level transformation; (ii) has both a structure and an orientation; and (iii) reduces an abundance of observations and experiences into role images.

3.4 Experiencing the Generic Roles Pattern (GRP) and the Generic Force-Field Pattern (GFP)

Until now, nothing has been said about the structure of the specific patterns or the role patterns. The image transformations reduce the number of patterns drastically and – when stable – the images can be used in a hierarchy of images. Similarly, role-image transformations do also reduce the number of role-patterns and of force-fields. But the numbers are still too large to obtain an intuitive experience, or an insight in the relations between these force-fields. Each addition of a dimension to a role in a role-pattern increases the complexity of the role-pattern by increasing the number of lines (dependent sets) that are neither minimal dependent, nor maximal independent in the full space (see the formula in A4). However, insight in the interdependence of force-fields can be obtained when we restrict ourselves to stable bipolar role force-fields. In that case a unique structure emerges. It has a wide applicability: any exchange implies a bipolar force-field; any institution implies generic role stability. So there is a plethora of generically stable, bipolar role force-fields in the world.

Definition 5: A Generic Bipolar Force-Fields Pattern *FF*(*World*)

A ternary Experience of some local phenomenon in the World is a stable role pattern of generically stable bipolar force-fields, *FF*(*World*), governing the specific local role pattern, *RP*(*World*), in the phenomenon and giving it an orientation. This governing structure of generic force-fields is a stable pattern that determines the relative position of each force-field, based on (i) the rule of motion within the structure and on (ii) the functions assigned to the spanning positions in the structure. One of these functions is the endogenous determination of values, which is performed by a generic bipolar force-field called *driving force-field*. It gives the local role pattern an orientation. The other spanning function is the endogenous determination of the *medium,* giving the local pattern the capability to incorporate values. □

The next question is: how does the structure of this $FF(World)$ looks like? What are the rules of motion within the structure? This problem is approached by repeating the method of limiting the scope of the projective geometries and reducing the dimension of the vector space. Starting with the specific positional geometry $G_{m,F}$, we obtained the role-specific geometry $G_{m,2}$. Going to the limit we arrive at the binary projective geometry $G_{2,2}$ represented in Figure 1. In this geometry, any line is a minimally dependent set in $G_{2,2}$ spanned by two points. (It is a bipolar role force field.) So lines are points in the dual space; both have the same number and the structure is self-dual. This structure is also a binary matroid representable over the projective geometry $PG(2,2)$, and called the Fano matroid.

This role pattern is self-dual, which property gives it the structure of a Fano matroid that determines the rules of motion between roles (see Figure 1). The structure $G_{2,2}$ contains seven points and seven lines. Any three independent points span the space. That is why the structure is called a *tripolar role pattern*. Any subsystem can be treated as a position in a higher-level system. The duality property introduced in the previous section states that with each position in the tripolar interaction structure, a unique subset – a hyperspace – can be associated. For example, $H(001)$ = {100, 110, 010}[11], $H(100)$ = {010, 011, 001}, and $H(111)$ = {101, 110, 011}. Inversely, we can associate with each dependent set in the interaction structure a position in the dual space, which space is isomorphic to the primal space. This operation may be repeated, which allows for identifying levels and moving between levels of a structure.

Definition 6: the Tripolar Role Pattern

Given the existence of a pattern of generically stable bipolar force-fields, $FF(World)$. The only structure that generates only stable bipolar force-fields, is the 2-dimensional projective space over the field {0,1}, denoted by $PG(2,2)$ or by $G_2 = (P, \oplus)$, with the rule of interaction defined by coordinate wise addition modulo 2. This structure, represented in Figure 1, is called the ***tripolar role pattern*** because it is spanned by three independent roles. The other four roles can be derived endogenously. The seven force fields are represented by lines in the diagram. It has also the properties of a specific matroid, called the Fano graph. □

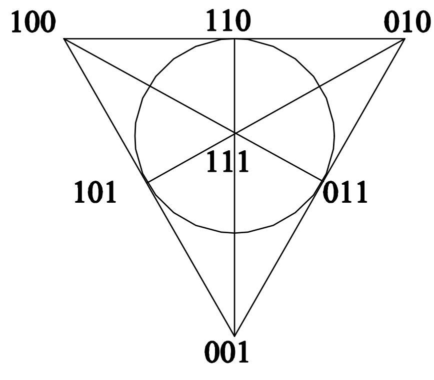

Figure 1 The tripolar role pattern

[11] One may check that 001 and 100 assign component wise the scalar 0 in: $(0\cdot 1) \oplus (0\cdot 0) \oplus (1\cdot 0) = 0$, and so forth.

In Figure 1 the positions 100, 010, and 001 are situated as extreme points. But any triple of positions that does not make a line or a circle may serve as spanning positions. For example, the relation {110, 011, 101} is dependent because 110 ⊕ 011 = 101, and form a hyperplane; the positions 101, 011 and 001 are independent and may serve as spanning positions. It may be noticed that the Fano matroid is not embedded in the 2-dimensional Euclidean plane, so a distance between the positions in Figure 1 or in the matroid is not defined.

The tripolar role pattern is the symmetric carrier of a theory of motion. The symmetry of that model is broken by specifying roles that generate forces, as is required for a theory of oriented motion. That theory of oriented motion recognizes the distinction between ends and means, or between a relative variety of possible realizations (values) to be produced and a relative uniformity in the resources (power) needed to produce the output. It does so by recognizing the difference in the structures that carry these images. An end (value) requires a pair of roles, because it describes the tension between a characteristic that is not present in some chosen position, but is present in some other position. The interaction between these two roles generates a value; together, the three roles form the driving subsystem of the phenomenon. A means consists of the resource that is – compared to the value to be realized – undifferentiated and performs in a single role, requiring only one position.

Claim 1: The Specific Implications of the Generic Role Pattern

The constructive theory asserts that each primary or secondary experience of a phenomenon in the world that involves stable bipolar force-fields, admits the Generic Role Assignment. Since the rules governing the generic roles also govern the specific roles, that is, both patterns are isomorphic, the specification of these roles and rules have to be consistent with the generic roles and rules. □

The secondary positional pattern, *RP*(*World*), may be embedded in an n-dimensional projective geometry over the field F_1; the ternary positional pattern, *FF*(*World*), is embedded in the 2-dimensional projective geometry over the field F_0, the tripolar role pattern.

4 Identifying Generic Roles in a Specific Role Pattern

4.1 *Defining roles and force-fields in the Generic Role Pattern*

On the basis of Claim 1, we can identify a specification of generic roles in a phenomenon in the world. That is made precise in the following definition.

Definition 7: The Generic Role Assignment T_3(*World*) to Specific Role Patterns

A generic role assignment assigns to a force field in some positional pattern, *P*(*World*), a role in the generic pattern, *FF* (*World*). □

Formally, he *generic role transformation* $T_3 : \Pi(n,F_1) \rightarrow \Pi(2,F_2)$, assigns to a subset of posi-tion sin the pattern *P*(*World*) a position in the generic role pattern described by $G_{2,2} = (P_2, \oplus)$. Similarly the *role-image transformation,* $T_3^* : \Pi(n,F_1) \rightarrow \Pi^*(2,F_2)$. The inverses of T_3 give a specification of the empowerment pattern required for generating an image or a force-field. This inverse gives the specific role assignment required for a generic role pattern. So when we compose the two transformations, we get the specific role assignment as it is governed by the generic role pattern. That map forms the basis for the governance of local patterns.

Definition 8: The Governance Frame: the Specific Role Assignment ∇(*World*) governed by the Generic Role Pattern

The composite assignment $\nabla = T_3^{-1}.\, T_3$, with ∇: *P*(*World)* → *P*(*World*), assigns a description of a pattern in the world, *P*(*World*), the local specification, to an existing pattern of positions in *P*(*World*), such that the rules governing the generic pattern are translated into rules governing the specific pattern. In particular, opposite roles in the *RP*(*World*) correspond to opposite roles in the *FF*(*World*) and the mediating roles correspond with the mediating role in the *FF*(*World*). □

Since the spanning positions in *FF*(*World*) are sufficient to derive the other positions in the *FF*(*World*), this subset of spanning positions is called a Frame and also denoted by ∇(*World*). The first application of this definition is giving characteristic names to the various roles and force-fields, based on the rules of motion ⊕ of the generic role pattern.

Step 1: Frame recognition; identifying the spanning capacities (roles) in ∇(*World*)

Each position is identified by the presence or absence of one of three characteristics in the phenomenon and defines (carries) a role.
The following three **spanning capacities (roles)** in ∇(*World*). ruled by (P, ⊕), are defined, see Figure 2; the subscript *i* indicates the presence of the first characteristic and the subscript –*i* the presence of the second characteristic:

- The *opposite values*, d_i and d_{-i}, in the positions 100 and 010;
- The *medium*, m, in the position 001;

Step 2: Identifying the three spanning force fields and dependent roles in ∇(*World*)

By means of the rules of motion (P,⊕), three spanning capacities define three force fields.

- The **driving force field:** the subsystem $D = \{d_i, d_{-i}, d\} = \{100,010,110\}$;
- The **embodying force fields:** the subsystems $M_i = \{d_i, b_i, m\} = \{100, 101, 001\}$ and $M_{-i} = \{d_i, b_{-i}, m\} = \{010, 011, 001\}$.

These force fields define three dependent **capacities or roles** in ∇(*World*). ruled by (P, ⊕), are then derived, see Figure 2, where the subscript *i* indicates the presence of the first characteristic and the subscript –*i* the presence of the second characteristic:

- The *opposite competences*. b_i and b_{-i}. in the positions 101 and 011;
- The *common value*, d, in the position 110;

The rules of interaction that govern the roles in an ∇(*World*) (Figure 2) are determined by the tripolar interaction structure, according to Property 2 and visualized in Figure 1. The different interactions between the various roles in an ∇(*World*) define the roles in Definition 4. For example, the opposite roles (d_i and d_{-i}) obey the rule 100 ⊕ 010 = 110 and create a common value (*d*); together they the driving subsystem (*D*). In this way, seven subsystems can be derived from the seven roles in the ∇(*World*). These subsystems or force fields constitute a dual structure of the ∇(*World*), in which each assumes a role again, as is illustrated in the right diagram of Figure 2. It may be observed that the names given to these subsystems in Definition 4 are interpretations of their performance. The performance is given and determined by the rules of behavior in an ∇(*World*). The naming of these subsystems is derived from their behavior and may be reinterpreted for specific situations. One subsystem of force fields has a fundamental significance, because it defines representation in the framework developed.

The diagram on the left in Figure 2 defines roles in the oriented interaction module ∇(*World*). The diagram on the right is a dual ∇(*World**) in which the positional roles correspond with force fields generated in the primal ∇(*World*).

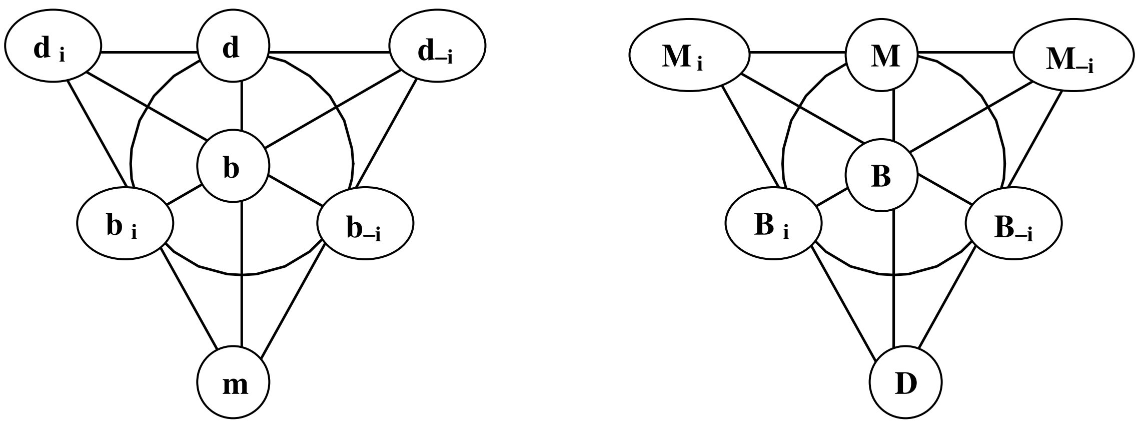

Figure 2. The positional capacities in the *Generic Role Pattern (GRP)*, respectively in the *Generic Force-Field Pattern (GFP),* which capacities correspond with force fields in *GRP*

First, we focus on the generic roles that constitute the various forces and capabilities a phenomenon according to Description level 3.

Step 3: Identifying the dependent Force Fields and the central capacity in ∇(*World*)

The last four force fields in the *FF*(*World*), each spanned by two of these roles, are:

- The **embodying force field:** the subsystem $M = \{d, b, m\}$ = {110.111.001};
- The **instrumental force fields:** the subsystems $B_i = \{d_i, b, b_{-i},\}$ = {100, 111, 011}, and $B_{-i} = \{d_{-i}, b, b_i\}$ = {010,111,101};
- The **balancing force field:** the subsystem $B = \{b_{-i}, b_i, d\}$ = {101, 011, 110}.

These force fields define the last capacity:

- The *balancing outcome*, b, in the position 111. □

Consider, for example, a ternary experience of en election, ∇(*Election*). Voters have opposite values for a social choice to be made, a president elect. The medium is a voting procedure through which voters can express their voice. These votes are opposite competences, which have to balance (by majority voting) to appoint the common value, i.e., the president elect. The *driving force field* consists of the interactions among the voters in terms of preferences, resulting in a common value. This force field of values is relatively independent from the medium, the election procedure. But voters receive competences or power from a given procedure to express their votes. This generates the *balancing force field*, in which the common value or the president elect is determined. This force field of competences is of course dependent on the medium, which is the institutional election procedure.

Next we focus on the force fields in the same a phenomenon and analyze the interaction between them by considering them as roles in the dual ORM.

Step 4: Identifying Incorporate Force Fields in ∇*(*World*) by (P*,⊕*)

- *The driving force field*, $D = \{d_i, d_{-i}, d\}$ in $FF(W)$ gets the medium position 001 in $\nabla^*(W)$;
- The *embodying or representational force fields*, $M_i = \{d_i, b_i, m\}$, $M_{-i} = \{d_{-i}, b_{-i}, m\}$, and $M = \{d, b, m\}$, in $FF(W)$ get the value positions 100, 010 and 011 in $\nabla^*(W)$
- The *instrumental force fields*, $B_i = \{d_i, b, b_{-i}\}$, and $B_{-i} = \{d^-_i, b, b_i\}$, in $FF(W)$ get the competence positions 101 and 011 in $\nabla^*(W)$
- The *balancing force field*, $B = \{b_{-i}, b_i, d\}$, in $FF(W)$ gets the outcome position 111 in $\nabla^*(W)$. □

Consider the election example again. The focus is now on the force fields, which fields result from role interactions by integration, aggregation or institutionalization. The embodying force fields of competences are the votes to be cast, which express some decision power, are the opposite values. The force field of social values is now the medium that gives a meaning to the votes and a criterion for the voting procedure to be imposed. The driving force field is therefore the interaction among competences, which results in a common competence or voting procedure. This is made conditional on the value system, resulting in the actual balancing force field of voting outcomes.

This example shows that the duality operation applied on a phenomenon in $\nabla(W)$inverses the focus from the value system (the driving force field) to the power system (the medium). The treatment of ends and means in a phenomenon is therefore symmetrical, although separated by a level of incorporation (integration).

4.2 Constructing a representation

The first step in the constructive theory is the description of the frame $\nabla(W)$ in $RP(W)$ of a phenomenon in the world, according to Definition 7. An artist, or a scientific community, looks at some phenomenon in the world from a specific angle by observing some force fields. This is the description phase that precedes the actual representation. Given the specification of the frame, the other components and rules of the specific role pattern $RP(W)$ can be derived. That results in $\nabla(W)$ and $\nabla^*(W)$ as described in the previous section.

Three force fields in $\nabla^*(W)$ have a particular relevance for this modeling research, viz., (D), (B), and the force field (M). The representational force field (M) is able to map the driving force field (D) into the balancing force field (B), or vice versa. So (D) and (B) are representations of each other under the representational force field (M). In functional notation: $M(D) = B$, or the inverse operation: $M^{-1}(B) = D$. In the sequel the field B is considered to be a representation of D under M.

Definition 8: The fundamental equation of representation

The interaction between the driving force field (D) and the representational force field (M) generates the balancing force field (B) in an ORM:

$$D \oplus M = B \tag{4.1}$$

This relation is called the fundamental equation of representation. □

The novelty of this definition is the identification and separation of the role of the representational force M in the representation problem. The factors in this equation are quite complex and usually too general for practical use. However, in the simplest example presented in Section 5.4, the force fields are optimization problems in the Euclidean space and the mediating force field M is simplified to a constant; then M reduces to a standard mathematical duality operation on sets. So there is a connection between representation and duality.

Claim 2: The representation claim

Consider a description $P(W)$ of a phenomenon in the world to which a balancing force field $\overline{B}$, a driving force-field D and a representational force-field M can be assigned Then $\overline{B}$ *represents* D, if $\overline{B} = M(D)$. □

The claim allows any phenomenon to be represented if only we accept a sufficiently weak representational force. An example is a nominal description. The number of specifications of both components may vary from one to infinity. The art of representing is to choose a frame with the strongest representational force. This goal is restricted by the available resources. Other examples are given in Section 5.

There are two ways to test the quality of the representation. Internal tests are based on the fact that each ORM requires a balance between its force fields, which can be checked after specification of the components of the ORM. For an ORM representing a real world phenomenon this includes empirical verification. So, the internal performance of an ORM may be checked by the following equations:

1. Check its opposite representations by specifying the equations:
$$d_i(M_i) = d_i(B_i) = d_i(D) \tag{4.2}$$
These are the learning equations.
2. Check the representational force on its social accord by specifying the equation:
$$m(M_i) = m(M_{-i}) = m(M) \tag{4.3}$$
These are the institutional resources.
3. Check its performance by specifying the instrumental forces:
$$b_i(M_i) = b_i(B_{-i}) = b_i(B) \tag{4.4}$$
These are the strategic variables.

The external validity of an ORM, the representation claim, is checked by substituting this ORM as driving force field in another ORM in which the description of the world it is supposed to represent is a balancing force field. The strength of the test depends on the degree of empirical elements that the representational force in this ORM contains in its specification.

4.3 The modular approach

A theory that is hierarchical in the sense that each pattern in that theory has the property that any part (role) of such a pattern may be viewed as a pattern itself is called a *modular theory*. This construction – called *the modular approach* – lets the intuitive experience about the rules of motion of phenomena in the world intact, as it may be applied on all levels of complexity. It supports the intuitive experience of both the relativity and the unity of all phenomena in the world, leaving the option of irrationality in an infinitely growing complex world open. This certainly is required for understanding and representing phenomena in the world.

Claim 3: The modular theory of representation

Given any primary or secondary experience of a phenomenon in the world, *RP*(*World*), that admits the Generic Role Assignment. Then any component of this pattern can be described as a module that also admits the Generic Role Assignment. □

It follows that a Generic Role Pattern is invariant for any change in the specification of a role or force field.

The generic Oriented Interaction Pattern results from a closure operation applied in Section 3.2 on the possibly infinite sequence in more complex patterns. The identification of a generic structure solves the problem of infinite regress, as identified by Aoki (2001), on a novel way. In this section that generic structure will be applied to develop a modular theory of representation, which allows for an infinite sequence of modules to represent the infinite history of phenomena in the world; see Diagram 1.

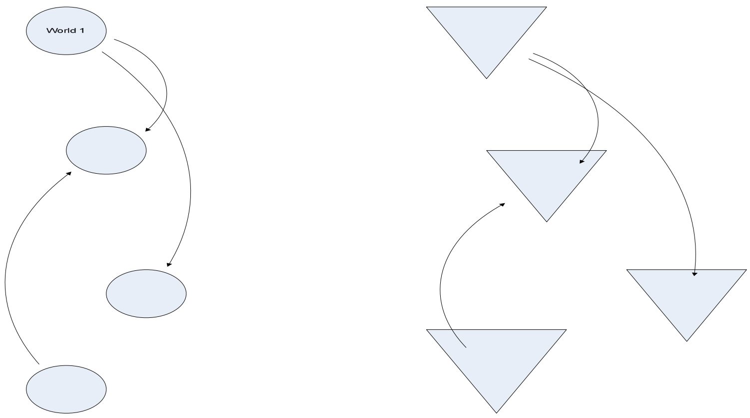

Diagram 1

Any phenomenon with an interaction structure contains subsystems that are internally interacting and externally interacting between each other. Any interacting structure is a module (subgroup) in an embedding interaction structure. Any role in an interacting structure is itself a module and contains an interaction structure. Another operation is focusing on a position in a given interaction structure. Since each phenomenon has an interaction structure, one is able to focus on the internal organization of some position, substituting the outcome into the given structure. This focusing process may be repeated.

Having derived a uniform tripolar interaction structure that is also a matroid, we can construct a network of modules, each module consisting of a tripolar interaction structure. A matroid is the mathematical tool that defines independency in a context and gives a structure to independent and dependent sets. In our context, a set is dependent if any position in that can be derived from a pair of other positions. That is the case for a hyperspace, which is called here a composition or subsystem. In a tripolar structure all positions are interrelated; some are directly related in a subsystem, other positions are indirectly related through a third position within the system.

This property will be applied on cases introduced in the next section. It allows for deriving a network of GRPs by subsequently enabling and applying a more comprehensive or a deeper representational force field *M*. This analytical property allows the designer to use the modular approach as a microscope or as a macro scope. A successful scientific discipline, for example, generates chains of modular models in its field.

This construction – called *the modular approach* – lets the intuitive experience about the rules of motion of phenomena in the world intact, as it may be applied on all levels of complexity. It supports the intuitive experience of both the relativity and the unity of all phenomena in the world, leaving the option of irrationality in an infinitely growing complex world open. This certainly is required for understanding and representing phenomena in the world.

Definition 9: The modular approach

Any phenomenon in the world may be split up in separable modules that are connected through consecutive application of the generic role assignment on its components. □

The art innovation consists of breaking up images by the driving force fields and integrating them by a higher level medium. Evolution is separating force fields into several new ones by replacing in the medium role outcomes that include integration mechanisms on a higher level.

5 Case studies

5.1 General equilibrium theory

Although neoclassical economic theory is constructed on essentially static concepts, it has not excluded economic forces in the theory producing economic change. The main concepts in the theory, such as production functions, preferences and resources, are exogenously given as primitive terms. These concepts may change over time, as in growth models, but if there is technological progress it is also determined exogenously. Time paths in multi-period models converge either to a static or oscillating equilibrium, or explode. Again, that is not the kind of motion we are looking for. The economic forces in general equilibrium theory have another character than the dynamics described in terms of the time dimension[12]. These are described by the following three classical models.

Market equilibrium (MEq)

The Marshallian partial equilibrium analysis envisions the market for one good. The question is how a price is formed that equilibrates demand and supply in one market.

World:

This model explains the working of the market mechanism in attaining an equilibrium price, which is a price (d) that equals demand (b^{-i}) to supply (b^{i}).

Frame:

This problem is framed in a two-dimensional Euclidean space, in which both demand and supply are quantified by real numbers, as well as the price. The driving subsystem specifies the marginal market reactions on a price increase, d^{i} ,d^{-i}, which are negative for demand and positive for supply, but not equal in absolute value. The medium integrates these derivatives into demand and supply functions. The institutional assumptions required are: a uniform price that performs as single determinant for supply and demand.

Performance:

The market reactions result in a decreasing demand function and an increasing supply function. (The individual instrumental subsystems are not activated.) The balancing subsystem

[12] This type of dynamic models fit of course also in the ORM framework, but requires another specification.

specifies price formation: the price change (d) is proportional to net demand, that is demand (b^{-i}) minus supply (b^{i}).
Evaluation:
This model is quite simple; it concerns the market for one commodity only, without explaining the behavior of the driving forces (the marginal market reactions), nor the effects of a price change from and to other markets. That has been achieved in the following model.

General equilibrium with individual choices (GMEq)

A more sophisticated model explains the working of the market mechanism from the individual demand and supply functions, which are extended to all commodity markets in the economy in order to allow for substitution effects between commodities. Without harming the argument, this extended model is restricted to an exchange economy and takes production of resources for granted.
World: problem = justum pretium
The world is a system of markets on which prices are formed that determine interdependent demand and supply of the commodities. The question is whether there exists a price that equilibrates all markets simultaneously.
Frame: = anonymous
The driving system contains a commodity space in R^n_+ and a price space equal to the simplex S^{n-1}. Each individual i has a demand function $f^i(p)$ and an individual resource vector, w^i, and is motivated to buy a commodity in excess demand on the market at the actual price and to sell when in excess supply. The medium aggregates the individual demands to a market demand: $f(p) = \Sigma f^i(p)$. This demand function is continuous and satisfies Walras law, $p(f(p)-w) = 0$, for all p in S^{n-1}, that is, the value of the excess demand is identically zero.
Performance:
The conditions mentioned above allow for invoking Kakutani Brouwer's fixed point theorem, which guarantees the existence of a vector of prices for all commodities at which the market demand equals the total resources available.
Evaluation:
This result is important, as it shows that individual choices may be consistent with available resources, but also economically empty, as no constructive procedure is given to attain such an equilibrium. Furthermore, institutional assumptions such as private ownership; absence of external effects; free access to all markets; voluntary exchange; these all and more are not identified in this frame. The following extension of the frame uncovers specific human motives as driving forces; this model has become the paradigm of neoclassical theory.

General equilibrium with individual preferences (GEq)

The focus of the second model of a general equilibrium goes deeper. The primitive concept is not the individual demand or supply, but the motivation of an anonymous demander or supplier in expressing demand and supply on the markets.
World:
The world is a system of markets on which prices are formed that determine interdependent demand and supply of the commodities. The question is whether there exists a price that equilibrates all markets simultaneously and such that all agents obtain the best possible bundles.
Frame:
The driving system is embedded in a commodity space and a price space, both in R^n_+. Each individual i maximizes preferences $\leq^i$ over a budget set, which is a function of prices and of his resources w^i. When these preferences are strictly convex, the individual demand functions $f^i(p)$ are continuous.

The medium contains the following common capabilities and restrictions. A system of competitive markets in the sense that no agent has an appreciable influence on price formation in the markets; private goods and private ownership; private use without external effects; a uniform price that performs as single determinant for supply and demand; the sum of the demands cannot exceed the sum of available resources.
Performance:
If the conditions mentioned above are satisfied, then there exists an equilibrium. This is a price vector at which all agents – each choosing their best possible consumption bundles – find a match for all commodities such that their demand equals supply and aggregate demand equals supply.
Evaluation:
This result has been extended to base producers' choices on input-output correspondences and profit-maximizing behavior. The resulting allocation is shown to be Pareto-optimal or allocative-efficient. This explains precisely Adam Smith's observation why citizens with motives based on self-interest unknowingly materialize the common good. The optimality criterion is a Nash-equilibrium in a non-cooperative game, which is almost void of institutional aspects; see Section 6.2 below. Many important institutional elements are not included in this General Equilibrium model. Some of the most prominent deficiencies have been detected by Keynes and are included in his macro-economic model.

5.2 The Keynesian model

When the GEq models give answers on price and allocation questions, they assume that all available productive resources will eventually be used. In a world of self-fulfilling private initiatives and a small government, this is a reasonable assumption. When, however, private initiatives become unpredictable and uncertain, when the size of government expenditure grows and when the economy opens to abroad, then the full use of all available productive capabilities (labor and capital) is not guaranteed any more and becomes a problem: the Keynesian theory of involuntary unemployment. Shackle (1961) describes it as follows: the business of enterprise involves investment in things whose outcome you cannot be certain of. When business is unsettled, they no longer know what the Marginal Value of a Product of an extra man is: it is nonexistent.
World:
In the balancing subsystem of the economy, actual purchasing power meets actual production power. If the purchasing power is too large, inflation results; if the production power is too large, unemployment results.
Frame:
The driving subsystem is an economy in which the purchasing abilities (that is total a priori domestic expenditures generated by consumption, investments, government expenditures and net exports) meet production abilities (or domestic income).
The medium is the whole complex of institutions that transform (purchasing and production) abilities into realizations or transactions. In the Keynesian model that means the market institutions are complemented with governmental fiscal and interest policy.
Performance:
The instrumental subsystems (B^i and B^{-i}) stabilize the movements when a purchasing component in (d^i), such as the government, interacts with production (b^{-i}) by anti-cyclical purchases or savings.
Evaluation:
In comparison with the neoclassical models, the Keynesian model extends the contents of the roles in the frame dramatically: it now also includes governmental institutions to monitor and

policy economic institutions. In the driving subsystem welfare policies are part of the purchasing abilities and infrastructure policies are part of the productive abilities. This is mirrored in the contents of the medium role, in which governmental institutions manage and realize these policies. The recent neo-institutional economics (NIE) has extended this development to non-governmental organizations.

5.3 Neo-institutional economics

North (1981, 2005) observes that the economic paradigm – neo-classical theory – was not created to explain the process of economic change. "The key to understanding the process of change is the intentionality of the players enacting institutional change and their comprehension of the issues. … Institutions are the constraints that human beings impose on human interaction. Those constraints … define the opportunity set in the economy." Neoclassical economic theory has overlooked the importance of both institutions and cognition, which North claims to be the two engines of change. They function as enhancing factors that simultaneously constrain, foster, and channel social change.

It may be true that neoclassical theory was not created to explain the process of change; it was, however, definitely created to influence institutions at the time and to restrict the role of existing governmental institutions. The 18th century economy could do the job – creating wealth – on its own and without protection even better! That confidence faded away in the 1930s, and since a balance between government and the market has been sought for. The innovative idea brought forward by neo-institutional economics is that institutions not only matter for performing governmental roles, but matter for any economic organization or social system, including private firms.

The modular theory of oriented motion is completely in line with this idea, specifying the two types of forces that frame and create change. The first is the 'driving force' from Definition 3, which represents "the intentionality of the players enacting institutional change", the second is the role of the 'medium', which consists of – among others – "institutions [as] the constraints that human beings impose on human interaction". Institutions – and the medium in general – not only perform as a constraint, but also as a capability to embody the values emerging from the driving subsystem. These ideas are expressed and specified in the following section.

5.4 Habermas' theory of communicative acting

Habermas (1981) develops a reconstructive philosophy in order to surpass self-reflection as a philosophical tool. *Rational reconstruction* is a method to surpass self-reflection and is based on a reflection of *die Lebenswelt*. Each actor has three types of rationality: (i) sense giving rationality, which is expressed by $M_i^{-1}(B_i) = D$, (ii) form giving rationality, expressed by $M_i (D) = B_i$, and (iii) instrumental rationality, which is expressed by $M_i (B_{-i}) = B$.

Communicative actions in D are exchanged in the sociability domain in B, called actualization: the balancing force field (B). The actions make use of a common resource, such as a language, and receive a meaning in the domain of subjectivity: the embodying force fields (M_i). The subjective expressions (Deutung), with speech intentions and speech acts as role forces resulting in an understanding: the embodying force fields (M_i).
"Any meaningful expression, be it an utterance, verbal or non-verbal, an artifact such as a tool or institution or scripture, can be bifocally identified as an observable event, and as an understandable objectivation or meaning. We certainly could describe, explain, or forecast a sound that is equivalent to a phonetic expression of a sentence, without having or getting any idea

about the meaning of this expression. In order to grasp (and fix) the meaning, one has to participate in some (actual or virtual) communicative actions in which the give utterance is used such that it becomes a comprehensible expression for the speakers, the listeners, and the members of that same language community."

The analysis of the normative intentions of actors occurs in the domain of intersubjectivity: the driving force field (*D*); analysis of the environment (Lebenswelt) in the balancing force field (*B*); and the objective productive (Strategie) in the instrumental force fields *(B_i)*.

5.5 Mathematical structures

The simplest duality relations are the formal ones as defined in mathematical models. Given a vector space P_n, its dual space P_n* contains the force fields (hyperplanes) from the first level as elements, and a corresponding dual operation ⊕* as defined in the Appendix. This process of transforming a pattern to a higher level of incorporation is called a *image transformation* in Definition 2, indicated by *.

Since the embodying subsystem (*M*) is in principle depending on many roles, its effects are hard to analyze unless it is reduced to a constant by fixing one component. That component cannot be the common value (*d*), because it is also present *D* and *B*. Fixing *b*, we reduce *M* to *m* and define:

(5.1)
$$m(D) = B \Leftrightarrow m(d_i, d, d_{-i}) = (b_i, d, b_{-i})$$
$$m^{-1}(B) = D \Leftrightarrow m^{-1}(b_i, d, b_{-i}) = (d_i, d, d_{-i})$$

The intuition behind these functions is that the m_+ function incorporates a driving force into a balancing one; it embodies ideas and meanings into a world. The m_- function disembodies or analyzes a balancing force in the world and gives its meaning or structure. These interpretations are derived from the orientation given in MOTOM. If one disregards this orientation, one obtains the mathematical duality image and both subsystems are just representations of each other. Two subsystems are perfect *representations* of each other if the duality operator transforms one subsystem into another and can regain the original subsystem without loss of information by applying the operation on the second subsystem. In mathematics these two subsystems are called self-dual under the duality operation. For example:

Linear Programming

The pair of dual models in Linear Programming, together with their inverses, forms a modular interaction structure. The balancing system (B) presents the input-output relations in terms of quantities. The driving subsystem (D) translates these production relations in terms of input-output prices. The inverse problems are presented horizontally; the dual problems vertically.

B: $\max \bar{p}x : A^i x \le \bar{y}$ and $\min \bar{q}y : A^{-i} y \ge \bar{x}$,

D: $\min q\bar{y} : qA^i \ge \bar{b}$ and $\max p\bar{x} : pA^{-i} \le \bar{q}$.

If some technical conditions are met, then both systems are dual representations of each other, as the duality operation is self-dual and the medium is only a simple scalar, 1.

m: $\bar{p}x = q\bar{y} = 1$ and $p\bar{x} = q\bar{y} = 1$

$$x,y \geq 0, \quad p,q \geq 0.$$

The balancing outcome is the pair of prices and quantities such that:

b: $$qA \cdot x = q \cdot Ax \quad \text{and} \quad pA^{-i} \cdot y = p \cdot A^{-i} y\,.$$

Similar mathematical representations exist between two domains (time and frequency): Fourier and Laplace transforms. The duality between (shadow)-prices and quantities is a standard tool in economics, see Diewert (1982).

Revealed preference

Given a decision maker's preferences and price-dependent resources, one is able to derive his or her choice as a function of prices. The question is whether the reverse problem can be solved: can one deduce a decision-maker's preferences from the choices he or she makes. The answer is given by revealed preference theory. Under suitable conditions – mainly convexity conditions on the preference structure – it is possible. So a duality correspondence between the preference space and the price (choice) space is established. The medium between the two spaces is the fixed amount of money that relates a bundle of commodities to some price vector and vice versa, by assuming that the agent chooses the best bundle available at a given price.
Revealed preference connects the demand functions in the GMEq-model with the individual preference in the GEq-model.

Game-theoretical worlds

A game describes strategic interdependencies between the players of the game. The rules of the game are fixed and have to be distinguished from the variable rules of agents' behavior, such as Nash-behavior.

World of a non-cooperative game:
The balancing subsystem consists of possible strategies or actions of one player in relation with the actions by other players.
Frame:
The driving subsystem consists of a set of players, each deriving individual utility (a pay-off) from an outcome, which outcome corresponds with a balancing vector of strategies.
The medium consists of a set of strategies; a strategy is a complete contingent plan or a decision rule for each player.
Performance:
The interaction between players is represented by the instrumental subsystems; each player decides whether it will change an action, given the actions of the other players. A Nash-equilibrium is attained if no player can improve upon a given composition of actions by the other players.

World of a cooperative game:
The balancing system consists of the choice of a player's membership for a possible coalition of players in relation to the membership choices made by other players for possible coalitions.
Frame:
The driving subsystem consists of a set of players, each deriving individual utility (a value) from an outcome, which outcome corresponds with membership of a balancing set of coalitions. The medium consists of the product-set of coalitions.
Performance:
The interaction between players is represented by the instrumental subsystems; each player decides whether it will change membership, given the actions of the other players. An equilib-

rium outcome is attained if no player can improve upon (block) a specific composition of coalitions.

5.6 *Enriching the medium: growth and imbalances*

When focusing on the analytical process, a mathematician tries to enrich the frame of a module so as to capture as many phenomena as possible: the process of generalization. Economists interpret the subsystems by adding institutional properties to the model, which is specification according to the mathematicians. However, the process of enriching the medium such that the model has a meaning in a wider context has also a generalizing aspect. The Keynesian model, for example, has government power besides market power and uses therefore a broader institutional medium than the GE-models. So enriching the medium such that it can cover more situations is a generalization.

Growth means that a result of motion within module, the specification of the balancing outcome (d) is used as medium (m) in the subsequent period. This is in line with the definition of growth in institutional economics.

Resolving imbalances requires a correct vision of the frame that can span balancing forces in a state of imbalance. American economists are used to take a narrow view of the medium that would be able to restore imbalances[13], such as a trade deficit. The classical economists advocate a movement from political mechanisms to market mechanisms, separating both domains. Keynes integrated both domains: from market mechanisms into political mechanisms. Today one observes integration of both domains again. Both history and dynamics can be analyzed by considering the medium as a capability and a constraint.

6 Conclusions and further vista

It is generally accepted that scientific representation needs to satisfy some plausible conditions, such as intentionality, informational equivalence (such as similarity) and capacities for functions (allowing for interpretations and inference)[14]. The constructive theory presented here satisfies these conditions. Intentionality is realized by the driving force-field; informational equivalence by the isomorphy between the two structures; the specific role assignment function (Definition 7) allows for empirical testing.

A second point is the relation between models and their interpretations; the relation between scientists and politicians. Model designers are giving interpretations that may be impressive, but simultaneously far from reality, so deceiving politicians and the media[15]. The art of economics is to restrict policy advices to the extent of analytical medium of their models. Neither does a perfect representation between model and the world exist in the social sciences, nor an all comprehensive module that can change the social world. The modular approach is not only more modest, but also more effective; it is therefore advisable.

[13] Martin Feldstein, for example, uses to conclude in his many presentations that the market mechanism alone is able to reverse the US-trade deficit and the decline of the dollar.
[14] These conditions have been mentioned by Chakravartty in a seminar in Tilburg, 11 October 2007.
[15] A notorious example is the interpretation of Arrow's impossibility theorem as "the end of democracy".

Frigg (2008) holds that every tenable account of model systems has to address the following issues. I add my comments with references in this paper to show that his conditions are met.

> "(I1) *Identity conditions*. Model systems are often presented by different authors in different ways. Nevertheless, many different descriptions are meant to describe the same model system. When are the model systems specified by different descriptions identical?"

In equation 4.3, this identity condition is formulated. It may also be considered as a social choice problem, which would be a separate problem and an embedded module.

> "(I2) *Attribution of properties*. In the previous sections I argued that model systems have concrete properties. How is this possible if model systems are not physical objects? What sense can we make of statements like 'the ball is charged' or 'the population is isolated from its environment' if there are no actual balls and populations?"

The role patterns in Definition 3 makes something actual that is not necessarily a physical entity.

> "(I3) *Comparative statements*. Comparing a model and its target system is essential to many aspects of modeling. We customarily say things like 'real agents are not behave like the agents in the model'. How can we compare something that does not exist with something that exists? Likewise, how are we to analyze statements that compare features of two model systems with each other like 'the agents in the first model are more rational than the agents in the second model'?"

The agents in the model may be representations of the real agents. The relevant properties for representation are described in the Medium role; when model and representation do not correspond, Claim 2, the representation claim is not satisfied.

> "(I4) *Truth in model systems*. There is right and wrong in a discourse about model systems. But on what basis are claims about a model system qualified as true or false, in particular if the claims concern issues about which our description of system remains silent? What we need is need is a account of truth in model systems, which, first, explains what it means for a claim about a model system to be true or false and which, second, draws the line between true and false statements at the right place (for instance, an account on which all statements about a model systems come out false would be unacceptable)."

The falsification of a claim is possible by showing inconsistencies in the interdependent generic force-fields. Usually the instrumental force-fields make imbalances visible.

> "(I5) *Epistemology*. We do investigate model systems and find out about them; truths about the model system are not concealed from us forever. But how do we find out about these truths and how do we justify our claims?"

Again, a model is a representation that has to satisfy the representation claim, which is not trivial.

Suárez (2004) formulates his Condition 1 as follows. *A* represents *B* only if the representational force of *A* points to *B*. When it is accepted that the representational force has the functional position of the Medium in the generic pattern, it is evident that my Representation Claim 2 satisfies his Condition 1.

In conclusion, I think a functional framework has been constructed that addresses the following aspects in the universe of phenomena:

- the intrinsically dynamic nature of phenomena in the universe;
- the unity of all phenomena;
- the separation and interrelation of all phenomena.

It gives a foundation for further research on imbalance management by determining whether a social phenomenon will expand or contract. Identifying and resolving imbalances when they are manageable is of utmost importance, because a society – as large as a country or as small as a household – may explode when social revolutions are not properly resolved.

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Appendix: On projective geometries and on matroids

The proper subject matter of a geometry is its invariant sets and the invariant functions on those sets (the Erlanger program). It follows that a property proven in one geometry is valid for all geometries. The transformation group *G* determines the character of the geometry, such as Euclidean, elliptic, hyperbolic or finite geometries, each of which may also use noninvariant sets or functions. It includes all the principal plane geometries (elliptic, parabolic, or hyperbolic) as sub-geometries. It does have coordinate systems, but they cannot be used to define a distance function.

In a projective geometry, there does not exist a zero or an identity position in the projective space; that allows for the intrinsically dynamic character of the interaction space. In contrast, the Euclidean space describes interactions as addition and multiplication and is suited for a world in which a zero-element exists that allows for reversible operations and operations on identical elements. This is needed in a mechanistic and probabilistic world. But the Euclidean space is not an appropriate tool to represent an interaction context. For example, when the addition of two identical distances has sense in a territory, an operation or interaction between two identical positions has no sense and is not defined.

Definition A1 A *geometry*

Let *S* be a nonempty set. A ***transformation group*** is a collection *G* of transformations $T{:}S \rightarrow S$ such that (i) *G* contains the identity; (ii) the transformations in *G* are invertible and their inverses are in *G*; and (iii) *G* is closed under composition.
A ***geometry*** is a pair (*S,G*) consisting of a nonempty set *S* and a transformation group *G*. The set *S* is the underlying space of the geometry. The set *G* is the transformation group of the geometry.
A ***figure*** is any subset *A* of the underlying set *S* of a geometry (*S,G*). Two figures *A* and *B* are ***congruent*** if there exists a transformation *T* in *G* such that *T*(*A*) = *B*, where *T*(*A*) is defined by the formula *T*(*A*) = {*Tz*: *z* is a point from *A*}.
Let *D* be a set of figures from (*S,G*). The set *D* is ***invariant*** if, for every member *B* of *D*, *T*(*B*) is also in *D*. So a set of figures is invariant if, for some figure B in the set, it contains all figures congruent to *B*.

Let *C* be the complex plane and *E* be the set of rigid motions, $Tz=e^{i\theta}z + b$, that preserve Euclidean distances between points. The pair (*C,E*) models the ***Euclidean geometry***.

Definition A2 A *field*

A ***field*** is a set *F* with two algebraic operations (addition and multiplication) that satisfy the following laws:

- (i) addition and multiplication are commutative and associative;
- (ii) *F* contains an additive identity (zero) and a multiplicative identity (one);
- (iii) every element of *F* has an additive inverse;
- (iv) every non-zero element of *F* has multiplicative inverse; and
- (v) addition and multiplication satisfy the distributive law: $a(b+c)=ab+ac$, for each *a,b* and *c* in *F*.

Definition A3 An *n-dimensional projective geometry over the field F*

Let (*F*, +, .) be a field and let *n* be a positive integer, representing the number of relevant characteristics (dimensions) needed for distinguishing a role in the interaction structure. Define

(3.1) $$P_n = \{(x_0, x_1, x_2, \ldots, x_n) : x_0, x_1, x_2, \ldots, x_n \text{ are in } F, \text{ not all zero}\}$$

to be the set of vectors with n+1 coordinates. Each point $p=(x_0,x_1,\dots,x_n)$ of P_n is identified with all scalar multiples kp, where k is any nonzero element of F. Let $GL[F,n]$ be the set of all invertible linear transformations of P_n.
The pair
(3.2) $$P_n[F]=(P_n,GL[F,n])$$
models an ***n-dimensional projective geometry over the field F***.

It induces a binary operation $\oplus_F$ on P_n, defined by:
(3.3) $$A\oplus_F B=C\Leftrightarrow C=A(B),\text{for } A\in P_n[F]$$
It may be noticed that P_n=$P_n[F]$, so any component in P_n is a linear transformation on P_n.

The projective space has the property that the zero-vector is not an element of this space. In this sense it differs from the Euclidean geometry that has an origin. This property characterizes a true interaction space in which all operations imply motion, as the zero-element would preclude motion, so define true rules of motion. The basic analytical assumption to the role approach made in Axiom 1 is that every interaction can be partitioned into a binary operation. Next we simplify the content of the relevant characteristics needed for distinguishing a role in the interaction structure. Assume that a characteristic cannot be described by a set of quantitative parameters, but only in qualitative terms of a property being present (1), or absent (0). Then the rules of interaction for each coordinate are: $0\oplus 0=1\oplus 1=0$ and $0\oplus 1=1\oplus 0=1$.

Definition A4 An *n-dimensional projective geometry over the field of integers modulo two*
In $PG(n,2)$, the n-dimensional projective space over F_2, the field over {0,1}, points are represented by {0,1}-vectors with n+1 coordinates. The operation adding is coordinate wise modulo 2.
Two independent points x and y determine one ***line***: $\{x, y, x\oplus y\}$.
Three independent points x, y and z (so $x\oplus y\neq z$) determine one ***plane***:
$\{x, y, z, x\oplus y, x\oplus z, y\oplus z, x\oplus y\oplus z\}$.
A ***hyperplane*** is a (n-1)-dimensional subspace and can be represented by the equation:
$$a_0x_0+a_1x_1+\dots+a_nx_n=0\Leftrightarrow(a_0,a_1\dots,a_n).$$
So there are as many hyperplanes as there are points, namely $2^{n+1}-1$. In general, if A(m,n)is the number of m-dimensional subspaces in PG(n,2), then the number of points (or hyperspaces) is:
$$A(m,n)=A(n-m-1,n)=\prod_{i=0}^{m-1}(2^{n+1-i}-1)\Big/\prod_{i=0}^{m-1}(2^{i+1}-1).$$
For example, the projective geometry PG(3,2) has $(2^4-1)/(2-1)$ = 15 points (or hyperspaces). It has $(2^4-1)(2^3-1)/(2-1)(2^2-1)$ = 35 lines. The projective geometry PG(2,2) has 7 points and 7 hyperplanes, which are equal to lines.

Definition A5 Duality and isomorphy
With each n-dimensional vector or position x in P_n, a unique subset in P_n called a **hyperplane** $H(x)=(p^0,p^1,\dots,p^n)$ with $p^i=(p_0^i,p_1^i,\cdots,p_n^i)$ can be associated that consists of the n+1 solutions of the scalar equation $x_0p_0\oplus x_1p_1\oplus\cdots\oplus x_np_n=0$. This hyperplane is also called a composition of positions in the vector space P_n. This composition of positions in P_n is repre-

sented as a vector (a position) in the dual interaction space P^*_n. An interaction operation $\oplus *$ on the composition space P^*_n defines **dual space** $G^*_n = (P^*_n, \oplus *)$ through the condition:
[x = y $\oplus$ … $\oplus$ z] implies [H(x) = H(y) $\oplus *$… $\oplus *$ H(z)], for all positions x, y and z in P_n.
Two algebraic structures $(P, \oplus)$ and $(R, \oplus)$ are **isomorphic** if there exists a one-to-one mapping g from P in R such that $g(a \oplus b) = g(a) \oplus g(b)$, for all elements a and b in P.

Definition A6 Vector matroids

A matroid is essentially a set with an 'independence structure' defined on it. A **matroid** *M* consists of a finite set *E* and a collection Φ of subsets (called *independent* sets) of *E* having the properties: (i) Any subset of an independent set is independent; (ii) Any two members of Φ which are maximal in a subset *S* of *E* (called *bases* of *M*) have the same cardinality. A subset *D* in *E* is called *dependent* if it is not independent. A minimal dependent set is called a *circuit*.
If *E* is a finite set of vectors in a vector space *V*, then we can define a *vector matroid* on *E* by taking as bases all linearly independent subsets of *E* which span the same subspace as *E*.

The focus in this paper is on dependent sets. The definition in terms of dependent sets is:
A matroid *M* is a pair (*E*, Δ), where *E* is a nonempty finite set, and Δ is a collection of subsets of *E* (called *circuits*) satisfying the properties: (i) No circuit properly contains another circuit; (ii) any two members of Δ which are minimal in a subset *S* of *E* have the same cardinality.

Given a matroid *M* on a set *E*, then the **matroid *M*** is **representable over a field F** if there exists a vector space *V* over *F* and a map φ from *E* to *V*, with the property that a subset *A* of *E* is independent in *M* if and only if φ is one-to-one on *A* and φ(*A*) is linearly independent in *V*. So *M* is isomorphic to a vector matroid defined in some vector space over *F*. A **binary matroid** is representable over the field of integers modulo two.

The **Fano matroid** *F* is the binary matroid defined on the set *E* = {1,0,0; 0,1,0; 0,0,1; 1,0,1; 0,1,1; 1,1,0; 1,1,1} whose bases are all those subsets of *E* with three elements except the subsets that lay on a line in Fig. 1.